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Now let T_3 be the time at which the wave front reaches another observation post O_3, then considering the data of two of the three posts at a time we obtain the equations

$$\begin{aligned} r_1 - r_2 &= V(T_1 - T_2), \\ r_1 - r_3 &= V(T_1 - T_3), \qquad (3) \\ \text{and} \quad r_2 - r_3 &= V(T_2 - T_3), \end{aligned}$$

which are the acoustic equations of three systems of circles and of as many hyperbolas. The origin of the wave is at the common center of three circles, Fig. 4, one from each system. Each of the three circles passes through one of the points O, is tangent to circles about the other two and has its center at the common point of intersection of the hyperbolas given by equations (3).

In general, if the times of arrival of a wave at n posts of known positions are given, $n(n-1)/2$ equations can be obtained from the data, each of which forms the equation of a system of acoustic circles. (The truth of this statement may be easily seen by considering the number of permutations and combinations of n things taken two at a time). Evidently there are two circles in each system whose centers are at the origin of the wave. Of the $n(n-1)$ circles having their centers at the origin of the wave $(n-1)$ pass through O_1 and consequently are coincident; $(n-1)$ pass through O_2 and are coincident; and so on to the $(n-1)$ which pass through O_n. Therefore in general there are n distinct circles in the $n(n-1)/2$ systems which have their centers at the origin of the wave. The circle which passes through any post O_i is tangent to a circle of radius $V(T_j - T_i)$ drawn about the post O_j as center. The tangency is external if $T_j > T_i$ and internal if $T_j < T_i$.

ON THE NUMERICAL VALUE OF i^i.

By H. S. UHLER, Yale University.

The primary object of the present note is to place on record the values of $e^{-\pi/2}$ and of seven related powers of the same base which I have recently calculated to more than fifty decimal places.

In order to avoid the use of any table of mathematical constants the following series was employed

$$y \equiv e^{\sin^{-1} x} = 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{5x^4}{4!} + \cdots \; [x^2 < 1]$$

$$= (1 + \sum_1^\infty t_{2k+1}) + (x + \sum_1^\infty t_{2k+2}).$$

Since the collections of series at my disposal did not contain a formula for the general term of the above series it was necessary to prove that

$$t_{2k+1} = (0^2+1)(2^2+1)(4^2+1) \cdots [(2k-2)^2+1]x^{2k}/(2k)!$$

and

$$t_{2k+2} = (1^2 + 1)(3^2 + 1)(5^2 + 1) \cdots [(2k-1)^2 + 1]x^{2k+1}/(2k+1)!,$$

where $k = 1, 2, 3, \cdots$. This was easily accomplished by the aid of Leibnitz's theorem in the manner illustrated in Williamson's *Differential Calculus*, 1889, pages 51–56; the fundamental equation being

$$(1 - x^2)\frac{d^{n+2}y}{dx^{n+2}} - (2n+1)x\frac{d^{n+1}y}{dx^{n+1}} - (n^2+1)\frac{d^n y}{dx^n} = 0.$$

For $x = \pm 1/2$, $y = e^{\pm\pi/6}$.

With $x = 1/2$ the terms involving even powers of x were calculated independently of the odd-power terms. Since, in both sets, each term was derived directly from the preceding one, the arithmetical operations involved had to be checked very carefully by the inverse operations, for a single incorrect digit would have vitiated some or all of the succeeding terms in the set in which the error was committed. The total number of even terms used was the same as that of the odd terms, namely 85. Each term was computed to the nearest unit in the fifty-fourth decimal place, 5×10^{-55} being counted as 1×10^{-54}. $e^{\pm(\pi/3)}$ was obtained by squaring $e^{\pm(\pi/6)}$, $e^{\pm(\pi/2)}$ by multiplying together $e^{\pm(\pi/3)}$ and $e^{\pm(\pi/6)}$, and $e^{\pm\pi}$ by squaring $e^{\pm(\pi/2)}$. Let σ_1 and σ_2 denote respectively the numerical values actually found for the sum of the even terms and of the odd terms respectively when $x = 1/2$. The results of the calculation were:

$$\begin{aligned}
\sigma_1 &= 1.14023\;83210\;76428\;79214\;11319\;80379\;35089\;07668\;97667\;51132\;0551,\\
\sigma_2 &= 0.54785\;34738\;88039\;80847\;57156\;47717\;43140\;33527\;83521\;77586\;1757,\\
e^{\frac{\pi}{6}} &= 1.68809\;17949\;64468\;60061\;68476\;28096\;78229\;41196\;81189\;28718\;2308,\\
e^{-\frac{\pi}{6}} &= 0.59238\;48471\;88388\;98366\;54163\;32661\;91948\;74141\;14145\;73545\;8794,\\
e^{\frac{\pi}{3}} &= 2.84965\;39082\;26361\;49747\;41273\;19852\;90439\;39640\;06102\;78112\;6866,\\
e^{-\frac{\pi}{3}} &= 0.35091\;98071\;78410\;96756\;57367\;15996\;95305\;83625\;73153\;62096\;1747,\\
e^{\frac{\pi}{2}} &= 4.81047\;73809\;65351\;65547\;30356\;66703\;83312\;63901\;70874\;66453\;4901,\\
e^{-\frac{\pi}{2}} = i^i &= 0.20787\;95763\;50761\;90854\;69556\;19834\;97877\;00338\;77841\;63176\;9614,\\
e^{\pi} &= 23.14069\;26327\;79269\;00572\;90863\;67948\;54738\;02661\;06242\;60021\;16,\\
e^{-\pi} &= 0.04321\;39182\;63772\;24977\;44177\;37171\;72801\;12757\;28109\;81063\;30854.
\end{aligned}$$

The reliability of the preceding numbers was tested by checks applied independently to the final results. Thus $s_1{}^2 - s_2{}^2 = 1$, where s_1 and s_2 are the exact values approximated above by the corresponding expressions in Greek letters, was used to check the values of those expressions. It is concluded that the calculated values of $e^{\pi/6}$ and $e^{-(\pi/6)}$ given above are certainly correct to 52 decimal places and they are probably in error by not more than two units in the fiftythird place. It was also found that to fifty-two places of decimals neither $e^{\pi} \cdot e^{-\pi}$ nor $e^{\pi/2} \cdot e^{-(\pi/2)}$ differed from unity.

In this MONTHLY, *1917*, 237, it is stated that—for the value of $e^{-(\pi/2)}$ or i^i—"Mr. Escott using Steinhauser's 20-place tables, gets .20787957635076190854687 while Professor Reynolds, using Hutton's 20-place tables, gets .2078795763-4917907781." Comparison of these numbers with my datum given above shows that Mr. Escott's differs from it by only about one unit in the twenty-second decimal place, whereas Professor Reynolds's number is discordant at, and beyond, the eleventh place. Since this may mean that there is an error in Hutton's tables it would be helpful if Professor Reynolds would investigate and report the cause of the discrepancy.

Finally, in the seventh revised edition of the *Smithsonian Physical Tables*, 1920, page 55, a short table of values[1] of $e^{(\pi/4)x}$, $e^{-(\pi/4)x}$ and their logarithms is given. If it should ever become necessary to compute tables of this kind to a greater number of decimal places than can be effected advantageously by logarithms, my data given above may be used either as basic numbers or as independent checks.

HISTORICAL NOTES ON THE RELATION $e^{-(\pi/2)} = i^i$.

BY R. C. ARCHIBALD, Brown University.

In 1719 Count Giulio Carlo de'Toschi di Fagnano showed, in effect, that the arc of the quadrant of a unit circle ($\pi/2$) is[2]

$$2 \log.(1 - \sqrt{-1})^{\frac{1}{2}\sqrt{-1}} \times (1 + \sqrt{-1})^{-\frac{1}{2}\sqrt{-1}}.$$

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[1] For $x = 1, 2, 3, \cdots 20$. The part of the table without logarithms is also given on page 91 of J. B. Dale, *Five Figure Tables*. London, 1903. The values of $e^{19\pi/4}$ differ materially in these two sources.—EDITOR.

[2] *Opere Matematiche del Marchese . . . de'Toschi di Fagnano*. Milano, volume 2, 1912. On page 406 we find "$\int \frac{dt}{1+t^2}$ esprime l'arco di cerchio [radius unity], la di cui tangente é t" [if the equations of the circle are $x = \cos\theta$ and $y = \sin\theta$, $t = \tan\theta$]. On pages 422–423 we find the following:

$$\int \frac{dt}{1+t^2} = \log.\ (1 - t\sqrt{-1})^{\frac{1}{2}\sqrt{-1}} \times (1 + t\sqrt{-1})^{-\frac{1}{2}\sqrt{-1}}$$

or

$$\text{“(8)} \qquad \int \frac{dt}{1+t^2} = \log.\ (A^2 - B^2).$$

[where $A = 1 + \frac{1}{2}t + \frac{1}{8}t^2 - \frac{7}{48}t^3 - \frac{43}{384}t^4 \cdots$, $B = \frac{1}{4}t^2\sqrt{-1} + \frac{1}{8}t^3\sqrt{-1} - \frac{3}{32}t^4\sqrt{-1} \cdots$].

"Queste due ultime equazioni manifestano una nuova, e bellissima proprietà del cerchio, ciascun arco del di cui quadrante à per suo elemento $\frac{dt}{1+t^2}$, quando la t denota la tangente dell'arco medesimo.

"Scolio IV.—Se l'arco di cerchio fosse equale al quadrante, allora la t diverrebbe infinita, e per avere il logaritmo equale al quadrante nulla gioverebbe l'equazione (8). In questo caso si divida per mezzo lo stesso quadrante, e la tangente dell'arco sudduplo di esso sarà eguale all'unità;